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Rigorous theory of nuclear fusion rates in a plasma

Lowell S Brown, David C Dooling and Dean L Preston

Los Alamos National Laboratory, Los Alamos, NM 87545, USA

E-mail: brownl@lanl.gov, dcd@lanl.gov and dean@lanl.gov

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Abstract
Real-time thermal field theory is used to reveal the structure of plasma corrections to nuclear reactions. Previous results are recovered in a fashion that clarifies their nature and new extensions are made. Brown and Yaffe have introduced the methods of effective quantum field theory into plasma physics. They are used here to treat the interesting limiting case of dilute but very highly charged particles reacting in a dilute, one-component plasma. The highly charged particles are very strongly coupled to this background plasma. The effective field theory proves that the mean field solution plus the one-loop term dominates; higher loop corrections are negligible even though the problem involves strong coupling. Such analytic results for very strong coupling are rarely available, and they can serve as benchmarks for testing computer models.

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1. General formulation

A nuclear reaction, which we schematically indicate by $1+2 \to 3+4$, takes place over a very short distance in comparison with particle separations in a plasma. Hence, it can be described by an effective local Hamiltonian density

$$\mathcal{H}(\mathbf{x}, t) = g\mathcal{K}(\mathbf{x}, t) + g\mathcal{K}^\dagger(\mathbf{x}, t). \tag{1}$$

The operator $\mathcal{K}$ describes, with interaction strength g, the destruction of the initial particles and the creation of the final particles; the operator $\mathcal{K}^\dagger$ does the reverse. Fermi's golden rule presents the rate as

$$\Gamma = \int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}Qt/\hbar} \int (\mathrm{d}^3\mathbf{x}) \langle \mathcal{K}^\dagger(\mathbf{x}, t)\mathcal{K}(0)\rangle_\beta. \tag{2}$$

The angular brackets $\langle \cdots \rangle_\beta$ denote the thermal average; Q is the reaction energy release.

The extension of imaginary time thermodynamic theory to include real-time behaviour was initiated long ago by Schwinger [1] and Keldysh [2]. When the particles entering into the

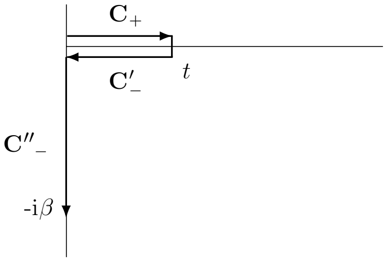

Figure 1. The $\mathbf{C}_+$ portion represents the interactions between the plasma and the final reaction particles that appear in $\langle \mathbf{x}, \mathbf{x}, t|\mathbf{0}, \mathbf{0}, 0\rangle_{3+4}^{V_C\phi}$. The $\mathbf{C}_-$ part is needed for the plasma interactions with the initial reaction particles that enter into $\langle \mathbf{0}, \mathbf{0}, -\mathrm{i}\beta\hbar|\mathbf{x}, \mathbf{x}, t\rangle_{1+2}^{V_C\phi}$. This contour has the real $\mathbf{C}'_-$ and purely imaginary $\mathbf{C}''_-$ parts.

nuclear reaction can be treated by Maxwell–Boltzmann statistics, *but with no other restrictions*, this real-time method shows that

$$\Gamma = g^2 \frac{n_1^{(0)} n_2^{(0)}}{\lambda_1^{-3}\lambda_2^{-3}} \int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}Qt/\hbar} \int (\mathrm{d}^3\mathbf{x}) \hat{Z}_C \left[\mathcal{V} \frac{\hbar}{\mathrm{i}} \frac{\delta}{\delta\phi} \right] \times \langle \mathbf{0}, \mathbf{0}, -\mathrm{i}\beta\hbar|\mathbf{x}, \mathbf{x}, t\rangle_{1+2}^{V_C\phi} \langle \mathbf{x}, \mathbf{x}, t|\mathbf{0}, \mathbf{0}, 0\rangle_{3+4}^{V_C\phi} \big|_{\phi=0}, \tag{3}$$

with the functional integral definition

$$\hat{Z}_C[\phi] = Z^{-1} \int \prod_b [\mathrm{d}\psi_b^* \,\mathrm{d}\psi_b] \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_C \mathrm{d}s\, L \right\} \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_C \mathrm{d}s \int (\mathrm{d}^3\mathbf{y}) \rho(\mathbf{y}, s) \phi(\mathbf{y}, s) \right\}. \tag{4}$$

All the field variables ψ in the plasma Lagrangian L and plasma charge density ρ are functions of the spatial coordinate $\mathbf{y}$, and the generalized time variable s runs along the contour $\mathbf{C}$ as shown in figure 1. The reacting particles have thermal wavelengths $\lambda_{1,2}$ and, with no plasma interactions, they would have number densities $n_{1,2}^{(0)}$.

The structure of the result (3) is easy to understand. The two transformation functions $\langle \cdots | \cdots \rangle^{V_C\phi}$ describe the propagation of the initial and final particles that undergo the nuclear reaction. The superscripts V_C indicate that these particles interact via their mutual Coulomb forces. The superscripts ϕ indicate that these particles also interact with an arbitrary external potential. The operator $\hat{Z}_C\left[\mathcal{V}\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta\phi}\right]$ produces the Coulomb interactions between the reacting particles and the background plasma.

In essentially all cases of interest, one can neglect the real-time portions $\mathbf{C}_+$ and $\mathbf{C}'_-$ because of the factor $\exp\{\mathrm{i}Qt/\hbar\}$: the relevant real-time scale $\hbar/Q$ is very much shorter than any characteristic plasma time. In many cases of interest, $\kappa r_{\max} \ll 1$, where κ is the Debye wave number and $r_{\max}$ is the turning point radius of the Coulomb interaction between the initial particles. Then the rate reduces to [3–5]

$$\Gamma = \Gamma_C \frac{N_1^{(0)}}{N_1} \frac{N_2^{(0)}}{N_2} \frac{N_{1+2}}{N_{1+2}^{(0)}}. \tag{5}$$

Here Γ_C is the nuclear reaction rate for a Maxwell–Boltzmann distribution of the initial $(1, 2)$ particles at temperature T but with no plasma background. The rate Γ_C contains the full effects of the Coulomb forces between the reacting particles, including quantum tunnelling (the Gamow factor). The number $N_a^{(0)}$ is the particle number obtained for a free gas grand canonical ensemble with chemical potential μ_a. The number N_a is the particle number of this species a with the same chemical potential μ_a but now interacting in the plasma. The subscripts $1 + 2$ denote a composite particle of charge $(Z_1 + Z_2)e$.

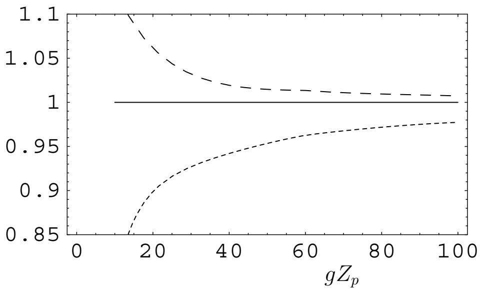

Figure 2. Ratios of $S[\mathrm{i}\phi_{\mathrm{cl}}] - Z_p$ for the ion sphere model result (short-dashed line) and the corrected ion sphere model (long-dashed line) to the exact numerical action.

2. Method illustrated by improving the ion sphere model

The simplest example has a weakly interacting one-component plasma, $g \ll 1$, where $g = \beta e^2\kappa/4\pi$. The effective field theory of Brown and Yaffe [6] shows that

$$N_p = \frac{N_p^{(0)}}{\mathcal{Z}} \int [\mathrm{d}\chi]\, \mathrm{e}^{-S[\chi]}, \tag{6}$$

where

$$S[\chi] = \int (\mathrm{d}^3\mathbf{r}) \left[\frac{\beta}{2}(\nabla\chi(\mathbf{r}))^2 - n(\mathrm{e}^{\mathrm{i}e\beta\chi(\mathbf{r})} - 1 - \mathrm{i}e\beta\chi(\mathbf{r})) - \mathrm{i}Z_p e\beta\delta(\mathbf{r})\chi(\mathbf{r}) \right]. \tag{7}$$

The normalizing partition function $\mathcal{Z}$ is defined by the functional integral whose action omits the δ function term in equation (7). The tree approximation is given by $S[\mathrm{i}\phi_{\mathrm{cl}}(\mathbf{r})]$ with

$$-\nabla^2\phi_{\mathrm{cl}}(\mathbf{r}) = en[\mathrm{e}^{-\beta e\phi_{\mathrm{cl}}(\mathbf{r})} - 1] + Z_p e\delta(\mathbf{r}). \tag{8}$$

This is the familiar Debye–Huckle form, but now placed in a systematic perturbative expansion where error can be ascertained. Including the one-loop correction gives

$$N_p = N_p^{(0)} \frac{\mathrm{Det}^{1/2}[-\nabla^2 + \kappa^2]}{\mathrm{Det}^{1/2}[-\nabla^2 + \kappa^2\,\mathrm{e}^{-\beta e\phi_{\mathrm{cl}}}]} \exp\{-S[\mathrm{i}\phi_{\mathrm{cl}}]\}. \tag{9}$$

We work in the limit where Z_p is so large that $gZ_p \gg 1$. The point charge $Z_p e/4\pi r$ part of $\phi_{\mathrm{cl}}(\mathbf{r})$ is large and dominates over a large range. This validates the Salpeter ion sphere model which approximates $[1 - \exp\{-\beta e\phi_{\mathrm{cl}}(\mathbf{r})\}] \simeq \theta(r_0 - r)$. The total plasma charge in this uniform sphere must cancel the impurity charge and so $r_0^3 = 3gZ_p/\kappa^3$. The first correction to the leading Salpeter solution can also be computed in analytic form except for a numerical integral. Including this correction gives, with $\mathcal{C} = 0.8498\ldots$,

$$-S[\mathrm{i}\phi_{\mathrm{cl}}] + Z_p \simeq \frac{3Z_p}{10}(3gZ_p)^{2/3}\left\{1 + \frac{10\mathcal{C}}{3gZ_p}\right\}. \tag{10}$$

Figure 2 compares this approximation to the action with its exact evaluation.

Brown and Yaffe [6] have shown that the one-loop correction for the background plasma with no impurity ions present is given by

$$\mathrm{Det}^{-1/2}[-\nabla^2 + \kappa^2] = \exp\left\{\int (\mathrm{d}^3\mathbf{r}) \frac{\kappa^3}{12\pi}\right\}. \tag{11}$$

In our limit the term $\kappa^2 \exp\{-\beta e\phi(\mathbf{r})\}$ in the one-loop determinant can be treated as being very slowly varying except when it appears in a final volume integral. Thus,

$$\frac{\mathrm{Det}^{1/2}[-\nabla^2 + \kappa^2]}{\mathrm{Det}^{1/2}[-\nabla^2 + \kappa^2\,\mathrm{e}^{-\beta e\phi_{\mathrm{cl}}}]} = \exp\left\{-\frac{\kappa^3}{12\pi}\frac{4\pi}{3}r_0^3\right\} = \exp\left\{-\frac{1}{3}gZ_p\right\}. \tag{12}$$

This result is physically obvious. The ion of high Z_p carves out a hole of radius r_0 in the original plasma. The original plasma is unchanged outside this hole. Corrections smooth out the sharp boundaries and produce only higher order terms. The original plasma had a vanishing electrostatic potential everywhere and the potential in the ion sphere picture now vanishes outside the sphere of radius r_0. Thus the thermodynamic potential of the plasma is reduced by the amount that was originally contained within the sphere of radius r_0, and this is exactly what is stated to one-loop order in equation (12). This argument carries on to the higher loop terms as well. A term involving n loops carries a factor g^n. The presence of the impurity modifies this to be $Z_p g^n$. With g sufficiently small, all the higher order loops make negligible contributions. The corrected impurity number N_p is hence given by equations (12) and (10) inserted into equation (9).

The number relation expresses the nuclear rate (5) in terms of the tree contribution. Including the first correction to the ion sphere result gives

$$\Gamma = \Gamma_C \exp\left\{\frac{3}{10}(3g)^{2/3}\left[(Z_1+Z_2)^{5/3} - Z_1^{5/3} - Z_2^{5/3}\right]\right\} \times \exp\left\{\left(\frac{9}{g}\right)^{1/3} \mathcal{C}\left[(Z_1+Z_2)^{2/3} - Z_1^{2/3} - Z_2^{2/3}\right]\right\}. \tag{13}$$

The first line agrees with the calculation of Salpeter [7]; the second is a new correction.

The number correction for the number of impurity ions N_p placed in the weakly coupled background plasma with number N can be used to construct the grand canonical partition function $\mathcal{Z}$ for the combined system by integrating the generic relation $N = \partial \ln \mathcal{Z}/\partial\beta\mu$. To simply bring out the main point, we now include only the leading terms. Standard thermodynamic relations then lead to the equation of state

$$pV = \left\{N - Z_p \frac{(3gZ_p)^{2/3}}{10} N_p\right\} T. \tag{14}$$

Although N_p/N may be small, there is a large pressure modification if Z_p is large.

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